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INVARIANTS OF INFINITE BLASCHKE PRODUCTS

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Dedicated to Professor Petru T. Mocanu

Abstract. The analytic continuation by symmetry with respect to the unit circle of infinite Blaschke products is studied and invariants of the restriction to some parts of the unit circle of these extended functions are obtained. Then analytic extensions of the respective invariants are constructed. The analogous results for infinite Blaschke products on the real projective plan are stated.

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1. EXTENSIONS OF INFINITE BLASCHKE PRODUCTS

Let (a_n) be a Blaschke sequence, i.e. a sequence of complex numbers such that $|a_n| < 1$ for every n and $\sum_{n=1}^{\infty}(1-|a_n|) < \infty$. A Blaschke factor is a Möbius transformation of the form:

$$b(z,a_n) = \frac{\overline{a}_n}{|a_n|}\frac{a_n - z}{1-\overline{a}_n z} \tag{1}$$

and an infinite Blaschke product is an expression of the form:

$$B(z) = \prod_{n=1}^{\infty} b(z,a_n). \tag{2}$$

It is known that for every Blaschke sequence (a_n) the corresponding Blaschke product converges uniformly on compact subsets of the open unit disk. This means that the sequence (B_n) of finite partial Blaschke products

$$B_n(z) = \prod_{k=1}^{n} b(z,a_k) \tag{3}$$

converges uniformly on compact subsets of the open unit disk D and $B(z) = \lim_{n\to\infty} B_n(z)$, $z \in D$. As every partial product $B_n(z)$ is a meromorphic function in $\overline{C}$, the question arises whether B could be extended outside D. A result due to Tanaka (see [6]) gives a partial answer to this question.

THEOREM 1. (Tanaka) *The following conditions are equivalent:*

(a) : $\sum_{n=1}^{\infty} \frac{1-|a_n|}{|\mathrm{e}^{\mathrm{i}\theta}-a_n|} < \infty.$

(b) : $\sum_{n=1}^{\infty} |\mathrm{e}^{\mathrm{i}\theta} - a_n| < \infty.$

(c) : B *converges absolutely at* $\mathrm{e}^{\mathrm{i}\theta}$.

Moreover, if these conditions are fulfilled, then:

$$\lim_{r \to 1_-} B(r\mathrm{e}^{\mathrm{i}\theta}) = B(\mathrm{e}^{\mathrm{i}\theta}). \tag{4}$$

In other words, Tanaka's theorem says that B can be extended by (4) at all the points $\mathrm{e}^{\mathrm{i}\theta} \in \partial D$ verifying the equivalent relation (a), (b), (c). We'll use this result in order to prove that, under some supplementary conditions on (a_n), much more can be said about the convergence of the sequence (B_n).

Let us notice first that if $\mathrm{e}^{\mathrm{i}\theta}$ is not a cluster point of (a_n) then there is $\delta > 0$ such that $|\mathrm{e}^{\mathrm{i}\theta} - a_n| \geq \delta$, $n = 1, 2, \ldots$. Consequently:

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\mathrm{e}^{\mathrm{i}\theta} - a_n|} \leq \frac{1}{\delta} \sum_{n=1}^{\infty} (1 - |a_n|) < \infty, \tag{5}$$

i.e., the condition (a) of Tanaka's theorem is fulfilled, and therefore B converges absolutely at $\mathrm{e}^{\mathrm{i}\theta}$.

Suppose now that $\mathrm{e}^{\mathrm{i}\theta_0}$ is not a cluster point of (a_n). Then, by a simple topological argument, there is an interval (α, β) such that $\theta_0 \in (\alpha, \beta)$ and every $\theta \in [\alpha, \beta]$ is not a cluster point of (a_n). Consequently, B converges absolutely at every point of the arc $\Gamma = \{z = \mathrm{e}^{\mathrm{i}\theta} : \theta \in [\alpha, \beta]\}$. Moreover, if $\theta, \theta' \in [\alpha, \beta]$ and $r < 1$, the inequality

$$\begin{aligned} |B(\mathrm{e}^{\mathrm{i}\theta}) - B(\mathrm{e}^{\mathrm{i}\theta'})| &\leq |B(\mathrm{e}^{\mathrm{i}\theta}) - B(r\mathrm{e}^{\mathrm{i}\theta})| + |B(r\mathrm{e}^{\mathrm{i}\theta}) - B_n(r\mathrm{e}^{\mathrm{i}\theta})| + \\ &+ |B_n(r\mathrm{e}^{\mathrm{i}\theta}) - B_n(r\mathrm{e}^{\mathrm{i}\theta'})| + |B_n(r\mathrm{e}^{\mathrm{i}\theta'}) - B(r\mathrm{e}^{\mathrm{i}\theta'})| + |B(r\mathrm{e}^{\mathrm{i}\theta'}) - B(\mathrm{e}^{\mathrm{i}\theta'})| \end{aligned} \tag{6}$$

shows that for $|\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta'}|$ small enough, the left hand side can be made as small as we want. Indeed, then we can choose n big enough and $1 - r$ small enough such that every term on the right hand side is as small as we want. This shows that the function B extended to Γ by the limit (4) is continuous on Γ.

The following is a slightly different form of Theorem 6.1, page 75, from [8]. Instead of Poisson integral formula, we are using as argument Theorem 1 and the direct analytic continuation theorem.

THEOREM 2. *If the set* E *of cluster points of* (a_n) *does not coincide with the whole unit circle* ∂D, *then the Blaschke product* B *can be extended by symmetry across the unit circle to a meromorphic function in* $\overline{C} - E$, *having as poles the points* $1/\overline{a}_n$.

Proof. Let us define $B^{\infty}(z)$ for every $z \in \overline{C}$, $|z| > 1$, by the formula:

$$B^\infty(z) = \frac{1}{\overline{B(1/\overline{z})}}, \text{ if } z \neq 1/\overline{a}_n \text{ and } B^\infty(1/\overline{a}_n) = \infty \tag{7}$$

and notice that since for every n, $B_n(z) = \frac{1}{\overline{B_n(1/\overline{z})}}$, $z \in \overline{C}$, we have that $B^\infty(z) = \lim_{n\to\infty} B_n(z)$, $|z| > 1$. Obviously, B^∞ is a meromorphic function for $|z| > 1$, having poles exactly at $1/\overline{a}_n$. Moreover, if $z_0 = \mathrm{e}^{\mathrm{i}\theta_0} \notin E$ we have $1/\overline{z}_0 = z_0$. Let us define $B^\infty(z_0)$ as $1/\overline{B(z_0)}$ and notice that $\overline{B^\infty(z_0)} = 1/\lim_{n\to\infty} B_n(z_0) = 1/B(z_0)$, in other words, $B^\infty(z_0) = 1/\overline{B(z_0)} = B(z_0)$. We see that the conditions of direct analytic continuation theorem across an arc $\Gamma \ni z_0$ of the unit circle (see [9], p. 183) are fulfilled. Such an arc Γ always exists, since E is a closed subset of the unit circle and therefore $\partial D - E$ is open in the trace topology of ∂D. Consequently, B and B^∞ are restrictions of a unique meromorphic function, which is analytic in a neighborhood of z_0. We use the same notation B for this extended function and we have that $B(z) = \lim_{n\to\infty} B_n(z)$ for every $z \in \overline{C} - E$, and the convergence is uniform on compact subset of the compliment of $E \cup \{1/\overline{a}_n : n = 1, 2, \ldots\}$ On the other hand there is no hope for a reasonable definition of B at the points of E, since if $\mathrm{e}^{\mathrm{i}\theta} \in E$, then there is a subsequence (a_{n_k}) such that $\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} 1/\overline{a_{n_k}} = \mathrm{e}^{\mathrm{i}\theta}$ and $B(a_{n_k}) = 0$, $B(1/\overline{a_{n_k}}) = \infty$, therefore $\lim_{z\to \mathrm{e}^{\mathrm{i}\theta}} B(z)$ does not exist.

The domain of B is a symmetric domain with respect to the unit circle and the function B is a symmetric function with respect to that circle in the sense that:

$$B(z) = \frac{1}{\overline{B(\frac{1}{\overline{z}})}}, \; z \in \overline{C} - E. \tag{8}$$

In other words, B has been extended, by using the symmetry principle, to $\overline{C} - E$. □

The function B^∞ can always be defined in terms of an infinite Blaschke product B, but if $E = \partial D$, B^∞ is not a direct continuation of B and we cannot talk about B as a meromorphic function in $\partial D - E$. Examples where $E = \partial D$ are cited in literature (see [7]), although unknown directly to us. We were therefore tempted to construct one. Let

$$a_{n,k} = (1 - 1/3^n)\mathrm{e}^{\mathrm{i}\frac{k\pi}{2^{n-1}}}, \; n = 1, 2, \ldots, \quad k = 1, 2, \ldots, 2^n. \tag{9}$$

It can be easily checked that:

$$\sum_{n=1}^{\infty} \sum_{k=1}^{2^n} (1 - |a_{n,k}|) = \sum_{n=1}^{\infty} \frac{2^n}{3^n} = 2, \tag{10}$$

therefore $(a_{n,k})$ is a Blaschke sequence and the corresponding Blaschke product

$$\prod_{n=1}^{\infty}\prod_{k=1}^{2^n} b(z,a_{n,k}) \tag{11}$$

converges uniformly on compact subsets of the open unit disc. On the other hand, every point θ of the interval $[0,2\pi]$ is a cluster point of the sequence

$$\left(\frac{k\pi}{2^{n-1}}\right),\ n=1,2,\dots,\quad k=1,2,\dots,2^n \tag{12}$$

and, as $\lim_{n\to\infty}(1-\frac{1}{3^n})=1$, every point $\mathrm{e}^{\mathrm{i}\theta}$ on the unit circle is a cluster point of $(a_{n,k})$, therefore $E=\partial D$.

2. INVARIANTS OF INFINITE BLASCHKE PRODUCTS

It is known (see [5]) that any finite Blaschke product B_n of degree n defines a $n-to-one$ self mapping of ∂D and the set G of continuous functions $U:\partial D\to\partial D$ such that $B_n\circ U=B_n$ on ∂D is a cyclic group of order n with respect to composition. The question arises whether similar properties of infinite Blaschke products exist. We expect the answer to this question to depend on the Blaschke sequence (a_n) and therefore we start with the simplest situation, namely when $a_n=r_n\mathrm{e}^{\mathrm{i}\theta}$. There is no loss of generality supposing $\theta=0$, i.e., $0\le a_n<1$ and $\sum_{n=1}^{\infty}(1-a_n)<\infty$. Particularly, $\lim_{n\to\infty}a_n=1$. Then, with $B_n(z)$ given by (3), we have

$$B(z)=\lim_{n\to\infty}B_n(z),\ z\neq 1 \tag{13}$$

and we know that $B(z)$ is a meromorphic function in $\overline{C}-\{1\}$, having the poles exactly at $\frac{1}{\overline{a_n}}=\frac{1}{a_n}$.

In order to describe the way B_n maps $n-to-one$ ∂D on itself, we need to solve an equation of the form $B_n(z)=1$, which is (since here $\overline{a_k}=a_k=|a_k|$):

$$\prod_{k=1}^{n}\frac{a_k-\ z}{1-a_k\ z}=1. \tag{14}$$

It is obvious that $z=-1$ is always a solution of the equation (14) and if n is even, then $z=1$ is also a solution of (14). Moreover, if $B_n(z_0)=1$, then $B_n(\overline{z_0})=1$.

We also can see that the equation (14) cannot have multiple solutions, since

$$\begin{aligned}B_n'(\mathrm{e}^{\mathrm{i}\theta})&=B_n(\mathrm{e}^{\mathrm{i}\theta})\sum_{k=1}^{n}\frac{a_k^2-1}{(a_k-\mathrm{e}^{\mathrm{i}\theta})(1-a_k\mathrm{e}^{\mathrm{i}\theta})}\\&=\mathrm{e}^{-\mathrm{i}\theta}B_n(\mathrm{e}^{\mathrm{i}\theta})\sum_{k=1}^{n}\frac{1-a_k^2}{|1-a_k\mathrm{e}^{\mathrm{i}\theta}|}\neq 0.\end{aligned} \tag{15}$$

Consequently, for every n, there is a partition: $0 = \theta_0^{(n)} < \theta_1^{(n)} < \cdots < \theta_{\lfloor \frac{n}{2} \rfloor}^{(n)} \leq \pi$ such that B_n maps every arc $\Gamma_k = \{z = \mathrm{e}^{\mathrm{i}(\pi-\theta)} : \theta_{k-1}^{(n)} \leq \theta < \theta_k^{(n)}\}$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ continuously and injectively on ∂D. The same is true for every arc $\Gamma_{-k+1} = \{z = \mathrm{e}^{\mathrm{i}(\pi+\theta)} : \theta_{k-1}^{(n)} < \theta \leq \theta_k^{(n)}\}$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. When $n = 2m$, then $\lfloor \frac{n}{2} \rfloor = m$ and $\theta_m^{(n)} = \pi$. When $n = 2m+1$, then besides the arcs Γ_k and Γ_{-k+1}, $1 \leq k \leq m$, there is also the arc $\Gamma_{m+1} = \{z = \mathrm{e}^{\mathrm{i}\theta} : -\pi + \theta_m^{(n)} < \theta \leq \pi - \theta_m^{(n)}\}$ which is mapped by B_n continuously and injectively on ∂D. Due to the continuity of B_n on the unit circle, there is a continuous passage from every mapping to the next one, with the convention that $\Gamma_{-\lfloor \frac{n}{2}+1 \rfloor}$ is next to $\Gamma_{\lfloor \frac{n}{2} \rfloor}$.

Let us write the equality (13) under the form:

$$B(z) = B_n(z)[1 + R_n(z)], \text{ where } \lim_{n\to\infty} R_n(z) = 0, \ z \neq 1. \tag{13'}$$

This shows that, if n is big enough, n of the roots of the equation $B(z) = 1$ will be slight perturbations of the roots of the equation $B_n(z) = 1$. Moreover, we can describe also the position of the remaining roots. Indeed, let's solve the equation $B_{n+1}(z) = 1$. We notice first that

$$B_{n+1}(z) = B_n(z)\frac{a_{n+1} - z}{1 - a_{n+1}z} = B_n(z)\left[1 + \frac{(1 - a_{n+1})(1 - z)}{1 - a_{n+1}z}\right], \tag{16}$$

where, due to the convergence of $\sum_{n=0}^{\infty}(1 - a_{n+1})$, we have that

$$\left|\frac{(1 - a_{n+1})(1 - z)}{1 - a_{n+1}z}\right| = o\left(\frac{1}{n}\right),$$

as $n \to \infty$. Again we can state that if n is big enough, the roots of the equation $B_{n+1}(z) = 1$ will be slight perturbations of the roots of the equation $B_n(z) = 1$, to which a new root is added. This last one should be $z_{n+1} = 1$, if n is odd, or the complex conjugate of the perturbation of the root $z_n = 1$ of the equation $B_n(z) = 1$, if n is even. This analysis suggest that the roots of (ζ_n) of the equation $B(z) = 1$ cannot accumulate to any point where $B(z)$ is analytic. Let us prove rigorously this affirmation. Suppose that ζ_0 is such a point and let ζ_{n_k} be such that $B(\zeta_{n_k}) = 1$ and $\lim_{k\to\infty} \zeta_{n_k} = \zeta_0$. Then, due to the continuity of $B(\zeta)$ at ζ_0, we have $B(\zeta_0) = 1$ and consequently

$$B'(\zeta_0) = \lim_{k\to\infty} \frac{B(\zeta_{n_k}) - B(\zeta_0)}{\zeta_{n_k} - \zeta_0} = 0,$$

which contradicts the relation (15). Therefore we can state the following Theorem 3.

Theorem 3. *Every Blaschke sequence of non-negative real numbers* (a_n) *determines a sequence* $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \pi$, $\lim_{n\to\infty} \theta_n = \pi$ *such that the corresponding Blaschke product maps continuously and injectively each* *one of the arcs* $\Gamma_n = \{z = \mathrm{e}^{\mathrm{i}(\pi-\theta)} : \theta_{n-1} \le \theta < \theta_n\}$, *as well as* $\Gamma_{-n+1} = \{z = \mathrm{e}^{\mathrm{i}(\pi+\theta)} : \theta_{n-1} < \theta \le \theta_n\}$, $n = 1, 2, \dots$ *on the unit circle. There is a continuous passage from every mapping to the next one in the sequence* (Γ_n), $n = \dots -1, 0, 1, \dots$

Now, let us remove the condition on a_n to be real and compare the Blaschke products $B_n(z) = \prod_{k=1}^{n} b(z, a_k)$ and $C_n(z) = \prod_{k=1}^{n} b(z, |a_k|)$. Suppose that $\mathrm{e}^{\mathrm{i}\alpha_1}, \mathrm{e}^{\mathrm{i}\alpha_2}, \dots, \mathrm{e}^{\mathrm{i}\alpha_n}$ are the roots of the equation $B_n(z) = 1$ and that $\mathrm{e}^{\mathrm{i}\beta_1}, \mathrm{e}^{\mathrm{i}\beta_2}, \dots, \mathrm{e}^{\mathrm{i}\beta_n}$ are the roots of the equation $C_n(z) = 1$. We only need to compare the last roots of the equations $B_{n+1}(z) = 1$ and $C_{n+1}(z) = 1$, since the others are slight perturbations of the former ones. This comes to evaluating the difference

$$\begin{aligned} b(z, a_{n+1}) - b(z, |a_{n+1}|) &= \frac{\overline{a_{n+1}}}{|a_{n+1}|} \frac{a_{n+1} - z}{1 - \overline{a_{n+1}} z} - \frac{|a_{n+1}| - z}{1 - |a_{n+1}| z} \\ &= \frac{1}{|a_{n+1}|} \left(\frac{a_{n+1} - z}{\frac{1}{\overline{a_{n+1}}} - z} - \frac{|a_{n+1}| - z}{\frac{1}{|a_{n+1}|} - z} \right) \\ &= \frac{1}{|a_{n+1}|} \frac{\left(|a_{n+1}| - \frac{1}{|a_{n+1}|} - a_{n+1} + \frac{1}{\overline{a_{n+1}}} \right) z}{\left(\frac{1}{\overline{a_{n+1}}} - z\right)\left(\frac{1}{|a_{n+1}|} - z\right)}. \end{aligned} \tag{17}$$

From this last expression it can be easily seen that $|b(z, a_{n+1}) - b(z, |a_{n+1}|) = o(\frac{1}{n})$, as $n \to \infty$. This implies that multiplying $B_n(z)$ by $b(z, a_{n+1})$ has similar effect on the roots of the equation $B_n(z) = 1$, as the (already known) effect of multiplying $C_n(z, |a_{n+1}|)$ by $b(z, |a_{n+1}|)$. Consequently, the Theorem 2.1 can be expressed in a more general setting, as follows:

THEOREM 4. *Suppose that the Blaschke sequence* (a_n) *converges to* $\mathrm{e}^{\mathrm{i}\theta_0}$. *Then there are infinitely many arcs* $\Gamma_n = \{z = \mathrm{e}^{\mathrm{i}\theta} : \theta_0 + \alpha_{n-1} \le \theta < \theta_0 + \alpha_n\}$, $n \in Z, \alpha_{-n} = -\alpha_n$, $0 = \alpha_0 < \alpha_1 < \dots$, $\lim_{n\to\infty} \alpha_n = \pi$, *which are mapped by the corresponding Blaschke product continuously and injectively on the unit circle. There is a continuous passage from every one of these mappings to the next one.*

We can now use the technique of [5] in order to prove the following theorem:

THEOREM 5. *If the Blaschke sequence* (a_n) *has a unique cluster point, then the set of continuous functions* $U : \partial D \to \partial D$ *such that* $B \circ U = B$ *on* ∂D *is an infinite cyclic group* G *with respect to the composition.*

Proof. Let us define as in [5] $\Psi_n : \partial D \to \Gamma_n$ such that $B(\Psi_n(\mathrm{e}^{\mathrm{i}\theta})) = \mathrm{e}^{\mathrm{i}\theta}$, $n \in Z$ and let $U_n : \partial D \to \partial D$ be defined as follows:

$$U_n|_{\Gamma_j} = \Psi_{n+j} \circ \Psi_j^{-1}. \tag{18}$$

Then $B \circ U_n|_{\Gamma_j}(\mathrm{e}^{\mathrm{i}\theta}) = B(\Psi_{n+j}(\Psi_j^{-1}(\mathrm{e}^{\mathrm{i}\theta}))) = \Psi_j^{-1}(\mathrm{e}^{\mathrm{i}\theta}) = B(\mathrm{e}^{\mathrm{i}\theta})$, for every $j \in Z$, therefore $B \circ U_n = B$ on ∂D, in other words U_n is an invariant of B. We need to show that the transformations U_n of ∂D form a cyclic group under composition. Indeed, U_0 is the identity and for any $m, n, j \in Z$, we have that U_{m+n} maps Γ_j on Γ_{m+n+j}. The same mapping can be obtained if we send Γ_j to Γ_{n+j} by U_n and then send Γ_{n+j} to Γ_{m+n+j} by U_m. In other words $U_{m+n} = U_m \circ U_n$. We skip the details, which are elementary. Let us denote by G the group generated in this way. It remains to show that G contains all continuous mappings V of ∂D on itself such that $B \circ V = B$.

We will use a similar argument to that employed in [5] in the finite case. Let us first notice that if $B(\mathrm{e}^{\mathrm{i}\theta}) = B(\mathrm{e}^{\mathrm{i}\theta'})$, then there is an n unique such that $\mathrm{e}^{\mathrm{i}\theta'} = U_n(\mathrm{e}^{\mathrm{i}\theta})$. Indeed, if $\mathrm{e}^{\mathrm{i}\theta} \in \Gamma_j$ and $\mathrm{e}^{\mathrm{i}\theta'} \in \Gamma_m$, then ζ and ζ' are uniquely determined, such that $\mathrm{e}^{\mathrm{i}\theta} = \Psi_j(\zeta)$ and $\mathrm{e}^{\mathrm{i}\theta'} = \Psi_m(\zeta')$, therefore $\zeta = B(\Psi_j(\zeta)) = B(\mathrm{e}^{\mathrm{i}\theta}) = B(\mathrm{e}^{\mathrm{i}\theta'}) = B(\Psi_m(\zeta')) = \zeta'$ and consequently $\mathrm{e}^{\mathrm{i}\theta'} = \Psi_m(\zeta') = \Psi_m(\zeta) = \Psi_m(\Psi_j^{-1}(\mathrm{e}^{\mathrm{i}\theta})) = U_{m-j}(\mathrm{e}^{\mathrm{i}\theta})$, i.e., $n = m - j$.

Now suppose that $V : \partial D \to \partial D$ is a continuous map such that $B \circ V = B$ and let us denote $F_j = \{\xi \in \partial D : V(\xi) = U_j(\xi)\}$ for every $j \in Z$. Due to the continuity of the functions involved, F_j are all closed subsets of ∂D. Since $B(V(\xi)) = B(\xi)$, by the previous remark, there is j such that $V(\xi) = U_j(\xi)$, therefore at least one of the sets F_j is not empty. Then, a connectedness argument implies that all the other sets F_k , $k \neq j$ are empty and $F_j = \partial D$, i.e. $V(\xi) = U_j(\xi)$ for every $\xi \in \partial D$, in other words $V = U_j$. This proves completely the theorem. □

3. THE CASE OF MULTIPLE CLUSTER POINTS OF THE SEQUENCE

Suppose that the sequence (a_n) has several cluster points forming a discrete set $\omega_1, \omega_2, \dots \in \partial D$. Then, an analysis similar to that in the previous section allows us to state the following conjecture. Between every two adjacent cluster points ω_k and ω_{k+1} there are infinitely many arcs $\Gamma_n^{(k)}, n \in Z$ accumulating exactly to ω_k when $n \to -\infty$ and ω_{k+1}, when $n \to +\infty$, such that B represents every arc $\Gamma_n^{(k)}$ continuously and injectively on ∂D. There is a continuous passage from the mapping of $\Gamma_n^{(k)}$ to that of $\Gamma_{n+1}^{(k)}$ for every $n \in Z$. Functions $\Psi_n^{(k)} : \partial D \to \Gamma_n^{(k)}$ can be defined as previously and also $U_n^{(k)} : \partial D \to \partial D$ by

$$U_n^{(k)}|_{\Gamma_j^{(m)}} = \Psi_{n+j}^{(k)} \circ [\Psi_j^{(m)}]^{-1},\ m \in Z. \tag{18'}$$

The functions $U_n^{(k)}$ form a group of invariants of B and for every k, any $U_n^{(k)}$ generate an infinite cyclic subgroup of G.

A similar construction is conceivable in an even more general situation, namely when the cluster points of the Blaschke product form a Cantor set on

∂D. Then the "removed" arcs will take the place of the arcs between ω_k and ω_{k+1}.

4. ANALYTIC CONTINUATION OF THE FUNCTIONS U_N

It is known (see [5]) that for a finite Blaschke product the functions U_n can be extended analytically to an open annulus symmetric with respect to ∂D. We can prove a similar result for infinite Blaschke products:

THEOREM 6. *Let K be a compact subset of $\partial D - E$. Then, there is an open neighborhood V of K (in C) such that every function U_n can be extended analytically to V. The extended functions still verify the equation $B \circ U_n = B$.*

Proof. As shown in Theorem 2, the function B is analytic in $C - E \cup A$, where $A = \{\frac{1}{\overline{a_n}} : n = 1, 2, \dots\}$. For every $z \in C - E \cup A$, the derivative of B is:

$$B'(z) = -B(z) \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{(a_n - z)(1 - \overline{a_n}\, z)}. \tag{19}$$

If $\zeta = \mathrm{e}^{\mathrm{i}\theta} \in \partial D - E$, then $(a_n - \zeta)(1 - \overline{a_n}\zeta) = -\zeta(a_n - \zeta)(\overline{a_n} - \overline{\zeta}) = -\zeta|a_n - \zeta|^2$, and $|B(\zeta)| = 1$, therefore

$$|B'(\zeta)| = \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|a_n - \zeta|^2} \quad > 0. \tag{20}$$

Consequently, the local inverse theorem (see [1], p.132) can be applied at the point ζ and we conclude that there is a neighborhood V_ζ of ζ, $V_\zeta \subset C - E \cup A$ such that B maps V_ζ conformally and topologically onto a region W_ζ. Therefore there is an analytic local inverse $\varphi_\zeta : W_\zeta \to V_\zeta$ of B. Let $\zeta_n^{(k)} \in \Gamma_n^{(k)}$, where $\Gamma_n^{(k)}$ are the arcs defined in the previous section and let $V_{\zeta_n^{(k)}}$, respectively $W_{\zeta_n^{(k)}}$ be the corresponding neighborhoods. Then we have:

$$\varphi_{\zeta_{n+j}^{(k)}} \circ B|_{\Gamma_j^{(m)}} = \Psi_{n+j}^{(k)} \circ [\Psi_j^{(m)}]^{-1} = U_n^{(k)}|_{\Gamma_j^{(m)}},\ m \in Z. \tag{21}$$

In other words, the function $U_n^{(k)}$ has the analytic extension $\varphi_{\zeta_n}^{(k)} \circ B$ in a neighborhood of $\zeta_n^{(k)}$. The set $\{V_\zeta : \zeta \in K\}$ represents an open covering of K. Since K is a compact set, there is a finite covering $\{V_{\zeta_1}, V_{\zeta_2}, \dots, V_{\zeta_p}\}$ such that (21) is true on $V = \cup_{j=1}^{p} V_{\zeta_j}$ and the theorem is completely proved. $\square$

5. INVARIANTS OF INFINITE BLASCHKE PRODUCTS IN P^2

It is known (see [3] and [4]) that Blaschke products can be defined also on the real projective plan P^2. These are projections on P^2 of ordinary symmetric Blaschke products in $\overline{C}$. A model of P^2 is obtained by the factorization $\overline{C}/ < h >$, where $< h >$ is the two element group generated by h. The symmetry of B means that B commutes with the antianalytic involution $h(z) = -1/\overline{z}$. A Blaschke product in P^2 is then a mapping $b : P^2 \to P^2$ defined by $b(\widetilde{z}) = \widetilde{B(z)}$, where $\widetilde{z} = \{z, h(z)\}$. If a_k is a zero of B, we'll say that $\widetilde{a_k}$ is a zero of b. If ζ is a cluster point of (a_n), we'll say that $\widetilde{\zeta}$ is a cluster point of $(\widetilde{a_n})$. Let $T = \{\widetilde{z} \in P^2 : z \in \partial D\}$. Then we can prove the following:

Theorem 7. *Let* b *be an infinite Blaschke product in* P^2 *whose zeros have a unique cluster point* $\widetilde{\zeta} \in T$. *Then the set of continuous functions* $u : T \to T$ *such that* $b \circ u = b$ *is an infinite cyclic group* $\widetilde{G}$ *with respect to composition.*

Proof. The Blaschke product b lifts to a unique analytic Blaschke product B in $\overline{C} - \{\zeta\}$ (see [2]). Let G be the group of invariants of B. For every $U_k \in G$, let us define $u_k : T \to T$ by $u_k(\widetilde{z}) = \widetilde{U_k(z)}$. Then for every $z \in \partial D$, $b \circ u_k(\widetilde{z}) = b(\widetilde{U_k(z)}) = \widetilde{B(U_k(z))} = \widetilde{B(z)}$, which shows that $u_k \in \widetilde{G}$. Vice-versa, given $u_k \in \widetilde{G}$, let us denote by U_k a lift of u_k to ∂D, i.e. a continuous function such that $\pi \circ U_k(z) = u_k \circ \pi(z)$ for every $z \in \partial D$. Then $\pi(B(U_k(z))) = b(\pi(U_k(z))) = b(u_k(\pi(z))) = b(\pi(z)) = \pi(B(z))$, for every $z \in \partial D$, which means that $B \circ U_k = B$, or $b \circ U_k \circ h = B$, in other words $U_k \in G$, or $U_k \circ h \in G$. By the previous section, it results that for every compact arc $K \subset \partial D$ such that $\zeta \notin K$, U_k or $U_k \circ h$ has an analytic extension to a neighborhood of K. As only one of U_k or $U_k \circ h$ can be analytic, the other being antianalytic, one and only one of the relationships $U_k \in G$, $U_k \circ h \in G$ is true. Suppose that we have denoted by U_k the analytic one, by an elementary reasoning it can be shown that the relationship $u_k(\widetilde{z}) = \widetilde{U_k(z)}$ is a group isomorphism and therefore $\widetilde{G}$ is an infinite cyclic group. □

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